

# Complete $r$ -partite subgraphs of dense $r$ -graphs

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## Abstract

Let  $r \geq 3$  and  $(\ln n)^{-1/(r-1)} \leq \alpha \leq r^{-3}$ . We show that:

Every  $r$ -uniform graph on  $n$  vertices with at least  $\alpha n^r/r!$  edges contains a complete  $r$ -partite graph with  $r-1$  parts of size  $\left\lfloor \alpha (\ln n)^{1/(r-1)} \right\rfloor$  and one part of size  $\left\lceil n^{1-\alpha^{r-2}} \right\rceil$ .

This result follows from a more general digraph version:

Let  $U_1, \dots, U_r$  be sets of size  $n$ , and  $M \subset U_1 \times \dots \times U_r$  satisfy  $|M| \geq \alpha n^r$ . If the integers  $s_1, \dots, s_{r-1}$  satisfy  $1 \leq s_1 \dots s_{r-1} \leq \left\lfloor \alpha^{r-1} \ln n \right\rfloor$ , then there exists  $V_1 \times \dots \times V_r \subset M$ , such that  $V_i \subset U_i$  and  $|V_i| = s_i$  for  $1 \leq i < r$ , and  $|V_r| > n^{1-\alpha^{r-2}}$ .

**Keywords:** *uniform hypergraph; number of edges; complete multipartite subgraph.*

In this note *graph* means  $r$ -uniform graph for some fixed  $r \geq 3$ .

Given  $c > 0$ , how large complete  $r$ -partite graphs must contain a graph  $G$  with  $n$  vertices and  $cn^r$  edges? This question was answered for  $r = 2$  in [1], and for  $r > 2$  in [2]:  $G$  contains a complete  $r$ -partite graph with each part of size  $a (\log n)^{1/(r-1)}$  for some  $a = a(c) > 0$ , independent of  $n$ .

Here we refine this statement for  $r \geq 3$  and extend it to digraphs. Letting  $K_r(s_1, \dots, s_r)$  be the complete  $r$ -partite graph with parts of size  $s_1, \dots, s_r$ , our most concise result reads as:

**Theorem 1** *Let  $r \geq 3$  and  $(\ln n)^{-1/(r-1)} \leq \alpha \leq r^{-3}$ . Every graph with  $n$  vertices and at least  $\alpha n^r/r!$  edges contains a  $K_r(s, \dots, s, t)$  with  $s = \left\lfloor \alpha (\ln n)^{1/(r-1)} \right\rfloor$  and  $t = \left\lceil n^{1-\alpha^{r-2}} \right\rceil$ .*

Theorem 1 follows immediately from a subtler one:

**Theorem 2** *Let  $r \geq 3$  and  $(\ln n)^{-1/(r-1)} \leq \alpha \leq r^{-3}$ . Let  $G$  be a graph with  $n$  vertices and at least  $\alpha n^r/r!$  edges. If the integers  $s_1, \dots, s_{r-1}$  satisfy  $1 \leq s_1 \dots s_{r-1} \leq \alpha^{r-1} \ln n$ , then  $G$  contains a  $K_r(s_1, \dots, s_{r-1}, t)$  with  $t > n^{1-\alpha^{r-2}}$ .*

It seems that a digraph setup is more natural for such results, e.g., Theorem 2 follows from

**Theorem 3** *Let  $r \geq 3$  and  $(\ln n)^{-1/(r-1)} \leq \alpha \leq r^{-3}$ . Let  $U_1, \dots, U_r$  be sets of size  $n$  and  $M \subset U_1 \times \dots \times U_r$  satisfy  $|M| \geq \alpha n^r$ . If the integers  $s_1, \dots, s_{r-1}$  satisfy  $1 \leq s_1 \dots s_{r-1} \leq \alpha^{r-1} \ln n$ , then there exists  $V_1 \times \dots \times V_r \subset M$  such that  $V_i \subset U_i$  and  $|V_i| = s_i$  for  $1 \leq i < r$ , and  $|V_r| > n^{1-\alpha^{r-2}}$ .*

We prove Theorem 3 by counting. For a better view on the matter we give a separate theorem, hoping that it may have other applications as well.

Let  $U_1, \dots, U_r$  be nonempty sets and  $M \subset U_1 \times \dots \times U_r$ , let the positive integers  $s_1, \dots, s_r$  satisfy  $|U_i| \geq s_i$  for  $1 \leq i \leq r$ . Write  $B_M(s_1, \dots, s_r)$  for the set of products  $V_1 \times \dots \times V_r \subset M$  such that  $V_i \subset U_i$  and  $|V_i| = s_i$  for  $1 \leq i \leq r$ .

**Theorem 4** *Let  $r \geq 2$ , let  $U_1, \dots, U_r$  be sets of size  $n$  and  $M \subset U_1 \times \dots \times U_r$  satisfy  $|M| \geq \alpha n^r$ . If*

$$2^r \exp\left(-\frac{1}{r}(\ln n)^{1/r}\right) \leq \alpha \leq 1$$

*and the integers  $s_1, \dots, s_r$  satisfy  $1 \leq s_1 \dots s_r \leq \ln n$ , then*

$$|B_M(s_1, \dots, s_r)| \geq \left(\frac{\alpha}{2^r}\right)^{rs_1 \dots s_r} \binom{n}{s_1} \dots \binom{n}{s_r}.$$

## Remarks

- The relations between  $\alpha$  and  $n$  in the above theorems need some explanation. First, for fixed  $\alpha$ , they show how large must be  $n$  to get valid conclusions. But, in fact, the relations are subtler, for  $\alpha$  itself may depend on  $n$ , e.g., letting  $\alpha = \ln \ln n$ , the conclusions are meaningful for sufficiently large  $n$ .
- Note that, in Theorems 1-3, if the conclusion holds for some  $\alpha$ , it holds also for  $0 < \alpha' < \alpha$ , provided  $n$  is sufficiently large.
- As Erdős showed in [2], most graphs with  $n$  vertices and  $(1 - \varepsilon) \binom{n}{r}$  edges have no  $K_r(s, \dots, s)$  for  $s \geq c(\log n)^{1/(r-1)}$  and sufficiently large constant  $c = c(\varepsilon)$ , independent of  $n$ . Hence, Theorems 1-3 are essentially best possible at least for fixed  $\alpha$ .
- Finally, observe that different relations hold for  $r = 2$ , e.g., the following version of Lemma 2 in [3] corresponds to Theorem 3:

*Let  $(\ln n)^{-1/2} \leq \alpha < 1/2$ , and let  $G$  be a bipartite 2-graph with parts of size  $n$  with at least  $\alpha n^2$  edges. Then  $G$  contains a  $K_2(s, t)$  with  $s = \lfloor \alpha^2 \ln n \rfloor$  and  $t > n^{1-\alpha}$ .*

## Proofs

First, some definitions.

Suppose  $U_1, \dots, U_r$  are nonempty sets and  $M \subset U_1 \times \dots \times U_r$ ; let the integers  $s_1, \dots, s_r$  satisfy  $0 < s_i \leq |U_i|$ ,  $1 \leq i \leq r$ .

Define  $M' \subset U_1 \times \dots \times U_{r-1}$  as

$$M' = \{(u_1, \dots, u_{r-1}) : \text{there exists } u \in U_r \text{ such that } (u_1, \dots, u_{r-1}, u) \in M\}.$$

For every  $R \in B_{M'}(s_1, \dots, s_{r-1})$ , let

$$N_M(R) = \{u : u \in U_r \text{ and } (u_1, \dots, u_{r-1}, u) \in M \text{ for every } (u_1, \dots, u_{r-1}) \in R\},$$

$$d_M(R) = |N_M(R)|.$$

For every  $v \in U_r$ , let

$$N_M(v) = \{(u_1, \dots, u_{r-1}) : (u_1, \dots, u_{r-1}, v) \in M\},$$

$$d_M(v) = |N_M(v)|,$$

$$D_M(v) = |\{R : R \in B_{M'}(s_1, \dots, s_{r-1}) \text{ and } v \in N_M(R)\}|.$$

Finally, for every integer  $s \geq 1$ , let

$$g_s(x) = \begin{cases} \binom{x}{s} & \text{if } x \geq s; \\ 0 & \text{if } x < s. \end{cases}$$

**Proof of Theorem 4** We use induction on  $r$ . Let first  $r = 2$ , and by symmetry assume that  $s_1 \geq s_2$ . Since  $g_{s_2}(x)$  is convex, we see that

$$\begin{aligned} |B_M(s_1, s_2)| &= \sum_{R \subset U_1, |R|=s_1} \binom{d_M(R)}{s_2} = \sum_{R \subset U_1, |R|=s_1} g_{s_2}(d_M(R)) \\ &\geq \binom{n}{s_1} g_{s_2} \left( \binom{n}{s_1}^{-1} \sum_{R \subset U_1, |R|=s_1} d_M(R) \right) \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \sum_{R \subset U_1, |R|=s_1} d_M(R) &= \sum_{u \in U_2} \binom{d_M(u)}{s_1} = \sum_{u \in U_2} g_{s_1}(d_M(u)) \geq n g_{s_1} \left( \frac{1}{n} \sum_{u \in U_2} d_M(u) \right) \\ &\geq n \binom{|M|/n}{s_1} \geq n \binom{\alpha n}{s_1}. \end{aligned}$$

We have

$$\alpha n \geq 4 \exp \left( \ln n - \frac{1}{2} (\ln n)^{1/2} \right) > 2 \exp \left( \frac{1}{2} \ln n \right) \geq 2 \ln n.$$

and so,  $\alpha n > 2s_1$ . Therefore,

$$n \binom{\alpha n}{s_1} \geq n \left( \frac{\alpha}{2} \right)^{s_1} \binom{n}{s_1},$$

and, since  $g_{s_2}(x)$  is non-decreasing, we obtain

$$|B_M(s_1, s_2)| \geq \binom{n}{s_1} g_{s_2} \left( n \binom{n}{s_1}^{-1} \binom{\alpha n}{s_1} \right) \geq \binom{n}{s_1} g_{s_2} \left( \left( \frac{\alpha}{2} \right)^{s_1} n \right).$$

Likewise, from

$$-\frac{1}{2} (\ln n)^{1/2} \leq \ln \frac{\alpha}{4} \leq \ln \frac{1}{4},$$

we see that  $n \geq e^{(\ln 16)^2}$ , and so,

$$(\alpha/2)^{s_1} n \geq (\alpha/2)^{\ln n} n = n^{1+\ln \alpha/2} \geq n^{0.3} \geq 2\sqrt{\ln n} \geq 2s_2.$$

This inequality implies that

$$\begin{aligned} |B_M(s_1, s_2)| &\geq \binom{n}{s_1} \binom{(\alpha/2)^{s_1} n}{s_2} \geq \alpha^{s_1 s_2} 2^{-s_1 s_2 - s_2} \binom{n}{s_1} \binom{n}{s_2} \\ &> \left(\frac{\alpha}{4}\right)^{s_1 s_2} \binom{n}{s_1} \binom{n}{s_2}, \end{aligned}$$

completing the proof for  $r = 2$ .

Assume now the assertion true for  $r - 1$ ; we shall prove it for  $r$ . We first show that there exist  $W \subset U_r$  and  $L \subset M$  with  $|L| > (\alpha/2) n^r$  such that  $d_L(u) \geq (\alpha/2) n^{r-1}$  for all  $u \in W$ . Indeed, apply the following procedure:

**Let**  $W = U_r$ ,  $L = M$ ;

**While** *there exists an  $u \in W$  with  $d_L(u) < (\alpha/2) n^{r-1}$*  **do**

*Remove  $u$  from  $W$  and remove all  $r$ -tuples containing  $u$  from  $L$ .*

When this procedure stops, we have  $d_L(u) \geq (\alpha/2) n^{r-1}$  for all  $u \in W$ . In addition,

$$|M| - |L| < (\alpha/2) n^{r-1} n \leq (\alpha/2) n^r,$$

implying that  $|L| \geq (\alpha/2) n^r$ , as claimed.

Since  $g_{s_r}(x)$  is convex, and

$$\sum_{R \in B_{L'}(s_1, \dots, s_{r-1})} d_L(R) = |L| = \sum_{u \in W} D_L(u),$$

we see that

$$\begin{aligned} |B_L(s_1, \dots, s_r)| &\geq \sum_{R \in B_{L'}(s_1, \dots, s_{r-1})} \binom{d_L(R)}{s_r} = \sum_{R \in B_{L'}(s_1, \dots, s_{r-1})} g_{s_r}(d_L(R)) \\ &\geq |B_{L'}(s_1, \dots, s_{r-1})| g_{s_r} \left( \frac{\sum_{R \in B_{L'}(s_1, \dots, s_{r-1})} d_L(R)}{|B_{L'}(s_1, \dots, s_{r-1})|} \right) \\ &= |B_{L'}(s_1, \dots, s_{r-1})| g_{s_r} \left( \frac{\sum_{u \in W} D_L(u)}{|B_{L'}(s_1, \dots, s_{r-1})|} \right) \end{aligned} \tag{1}$$

On the other hand  $s_1 \cdots s_{r-1} \leq s_1 \cdots s_r \leq \ln n$ . Also, for every  $u \in W$ , we have

$$\frac{d_L(u)}{n^{r-1}} \geq \frac{\alpha}{2};$$

hence, in view of

$$\frac{\alpha}{2} \geq 2^{r-1} e^{-\sqrt[r]{\ln n}/r} > 2^{r-1} e^{-r^{-1} \sqrt[r]{\ln n}/(r-1)},$$

we can apply the induction hypothesis to the sets  $U_1, \dots, U_{r-1}$ , the numbers  $s_1, \dots, s_{r-1}$ , and the set  $N_L(u) \subset U_1 \times \cdots \times U_{r-1}$ . We obtain

$$D_L(u) \geq \left(\frac{\alpha/2}{2^{r-1}}\right)^{(r-1)s_1 \cdots s_{r-1}} \binom{n}{s_1} \cdots \binom{n}{s_{r-1}}$$

for every  $u \in W$ . This, together with  $|W| \geq |L|/n^{r-1} \geq \alpha n/2$ , gives

$$\sum_{u \in W} D_L(u) \geq \frac{\alpha n}{2} \left(\frac{\alpha}{2^r}\right)^{(r-1)s_1 \cdots s_{r-1}} \binom{n}{s_1} \cdots \binom{n}{s_{r-1}}.$$

Note that the function  $g_{s_r}(x/k)k$  is non-increasing in  $k$  for  $k \geq 1$ . Hence, from

$$|B_{L'}(s_1, \dots, s_{r-1})| \leq \binom{n}{s_1} \cdots \binom{n}{s_{r-1}}$$

and (1), we obtain

$$\begin{aligned} |B_L(s_1, \dots, s_r)| &\geq \binom{n}{s_1} \cdots \binom{n}{s_{r-1}} g_{s_r} \left( \binom{n}{s_1}^{-1} \cdots \binom{n}{s_{r-1}}^{-1} \sum_{u \in W} D_L(u) \right) \\ &\geq \binom{n}{s_1} \cdots \binom{n}{s_{r-1}} g_{s_r} \left( \frac{\alpha}{2} \left(\frac{\alpha}{2^r}\right)^{(r-1)s_1 \cdots s_{r-1}} n \right). \end{aligned} \quad (2)$$

To continue we need the following

**Claim 5** *The condition*

$$2^r \exp\left(-\frac{1}{r}(\ln n)^{1/r}\right) \leq \alpha \leq 1$$

*implies that*

$$\frac{\alpha}{2} \left(\frac{\alpha}{2^r}\right)^{(r-1)s_1 \cdots s_{r-1}} n \geq 2s_r.$$

*Proof* We have

$$\frac{\alpha}{2} \left(\frac{\alpha}{2^r}\right)^{s_1 \cdots s_r} \geq \frac{\alpha}{2} \left(\frac{\alpha}{2^r}\right)^{\ln n} > e^{-\sqrt[r]{\ln n}/r} \left(e^{-\sqrt[r]{\ln n}/r}\right)^{\ln n} = (en)^{-\sqrt[r]{\ln n}/r}. \quad (3)$$

On the other hand

$$e^{\ln 4 - \sqrt[r]{\ln n}/2} = 2^2 e^{-\sqrt[r]{\ln n}/2} \leq 2^r e^{-\sqrt[r]{\ln n}/2},$$

and so,  $\ln n \geq (\ln 16)^2$ , implying in turn that  $n \geq e^{(\ln 16)^2} = 16^{\ln 16}$ . Routine calculus shows that  $n^{1/2} - 4 \ln n$  increases for  $n \geq 16^{\ln 16}$ , and so,

$$n^{1/2} - 4 \ln n \geq (16^{\ln 16})^{1/2} - 4 \ln 16 > 0.$$

Now, from (3) we obtain

$$\begin{aligned} \frac{\alpha}{2} \left(\frac{\alpha}{2^r}\right)^{s_1 \cdots s_r} &\geq (en)^{-\sqrt[r]{\ln n}/r} > (4n)^{-(\ln n)/2} = \left(\frac{1}{2n^{1/2}}\right)^{\ln n} \\ &> \left(\frac{2 \ln n}{n}\right)^{\ln n} \geq \left(\frac{2s_r}{n}\right)^{s_r}, \end{aligned}$$

completing the proof of the claim. □

From (2) and the definition of  $g_{s_r}(x)$  we see that

$$\begin{aligned} |B_L(s_1, \dots, s_r)| &\geq \binom{n}{s_1} \cdots \binom{n}{s_{r-1}} \left( \frac{\alpha}{2} \left( \frac{\alpha}{2^r} \right)^{(r-1)s_1 \cdots s_{r-1}} \right)^{s_r} \binom{n}{s_r} \\ &\geq \left( \frac{\alpha}{2} \right)^{s_r} \left( \frac{\alpha}{2^r} \right)^{(r-1)s_1 \cdots s_r} \binom{n}{s_1} \cdots \binom{n}{s_r} \\ &> \left( \frac{\alpha}{2^r} \right)^{rs_1 \cdots s_r} \binom{n}{s_1} \cdots \binom{n}{s_r}, \end{aligned}$$

completing the proof.  $\square$

**Proof of Theorem 3** As in the proof of Theorem 4 we find  $W \subset U_r$  and  $L \subset M$  with  $|L| > (\alpha/2) n^r$  such that  $d_L(u) \geq (\alpha/2) n^{r-1}$  for all  $u \in W$ . Let

$$t = \max \{d_L(R) : R \in B_{L'}(s_1, \dots, s_{r-1})\}.$$

We have

$$|B_{L'}(s_1, \dots, s_{r-1})| \leq \binom{n}{s_1} \cdots \binom{n}{s_{r-1}},$$

and so

$$t \binom{n}{s_1} \cdots \binom{n}{s_{r-1}} \geq t |B_{L'}(s_1, \dots, s_{r-1})| \geq |L| = \sum_{u \in W} D_L(u). \quad (4)$$

To continue we need the following

**Claim 6** *The condition  $(\ln n)^{-1/(r-1)} \leq \alpha \leq r^{-3}$  implies that*

$$2^{r-1} \exp \left( -\frac{1}{r-1} (\ln n)^{1/(r-1)} \right) \leq \frac{\alpha}{2} \leq 1$$

*Proof* The upper bound is obvious, so we have to prove that

$$\ln \frac{\alpha}{2^r} \geq -\frac{1}{r-1} (\ln n)^{1/(r-1)}.$$

The function  $x^x$  decreases for  $0 < x < e^{-1}$ , and  $\alpha \leq r^{-3}$ ; hence

$$\alpha \ln \frac{\alpha}{2^r} \geq \frac{1}{r^3} \ln \frac{1}{r^3 2^r} = -\frac{1}{r^3} (3 \ln r + r \ln 2) > -\frac{3r}{r^3} \geq -\frac{1}{r}, \quad (5)$$

and so,

$$\ln \frac{\alpha}{2^r} > -\frac{1}{(r-1)\alpha} \geq -\frac{1}{r-1} (\ln n)^{-1/(r-1)},$$

completing the proof of the claim.  $\square$

Since for every  $u \in W$  we have

$$\frac{d_L(u)}{n^{r-1}} \geq \frac{\alpha}{2},$$

in view of Claim 6, we may apply Theorem 4 to the sets  $U_1, \dots, U_{r-1}$ , the numbers  $s_1, \dots, s_{r-1}$ , and the set  $N_L(u) \subset U_1 \times \dots \times U_{r-1}$ , thus obtaining

$$D_L(u) \geq \left(\frac{\alpha/2}{2^{r-1}}\right)^{(r-1)s_1 \dots s_{r-1}} \binom{n}{s_1} \dots \binom{n}{s_{r-1}}$$

for every  $u \in W$ . This, together with  $|W| \geq |L|/n^{r-1} \geq \alpha n/2$ , gives

$$\sum_{u \in W} D_L(u) \geq \frac{\alpha n}{2} \left(\frac{\alpha}{2^r}\right)^{(r-1)s_1 \dots s_{r-1}} \binom{n}{s_1} \dots \binom{n}{s_{r-1}}.$$

Substituting this bound in (4), we find that

$$t \geq \frac{\alpha}{2} \left(\frac{\alpha}{2^r}\right)^{(r-1)s_1 \dots s_{r-1}} n \geq \frac{\alpha}{2} \left(\frac{\alpha}{2^r}\right)^{(r-1)\alpha^{r-1} \ln n} n > \left(\frac{\alpha}{2^r}\right)^{r\alpha^{r-1} \ln n} n.$$

Finally, (5) gives

$$\left(\frac{\alpha}{2^r}\right)^{r\alpha^{r-1} \ln n} > e^{-\alpha^{r-2} \ln n} = n^{-\alpha^{r-2}},$$

completing the proof of Theorem 3. □

**Proof of Theorem 2** Suppose  $r, \alpha, n$ , and  $G$  satisfy the conditions of the theorem. Let  $U_1, \dots, U_r$  be  $r$  copies of the vertex set  $V$  of  $G$ , and let  $M \subset U_1 \times \dots \times U_r$  be the set of  $r$ -vectors  $(u_1, \dots, u_r)$  such that  $\{u_1, \dots, u_r\}$  is an edge of  $G$ . Clearly,  $|M| \geq r! (\alpha n^r / r!) = \alpha n^r$ . Theorem 3 implies that there exists a set  $V_1 \times \dots \times V_r \subset M$  such that  $V_i \subset V$  and  $|V_i| = s_i$  for  $1 \leq i < r$ , and  $|V_r| > n^{1-\alpha^{r-2}}$ . Note that the sets  $V_1, \dots, V_r$  are disjoint, for the edges of  $G$  consist of distinct vertices. Hence  $V_1, \dots, V_r$  are the vertex classes of an  $r$ -partite subgraph of  $G$  with the desired size. □

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